A Strengthening of Erdős-Gallai Theorem and Proof of Woodall's Conjecture

Binlong Li

Joint work with Bo Ning

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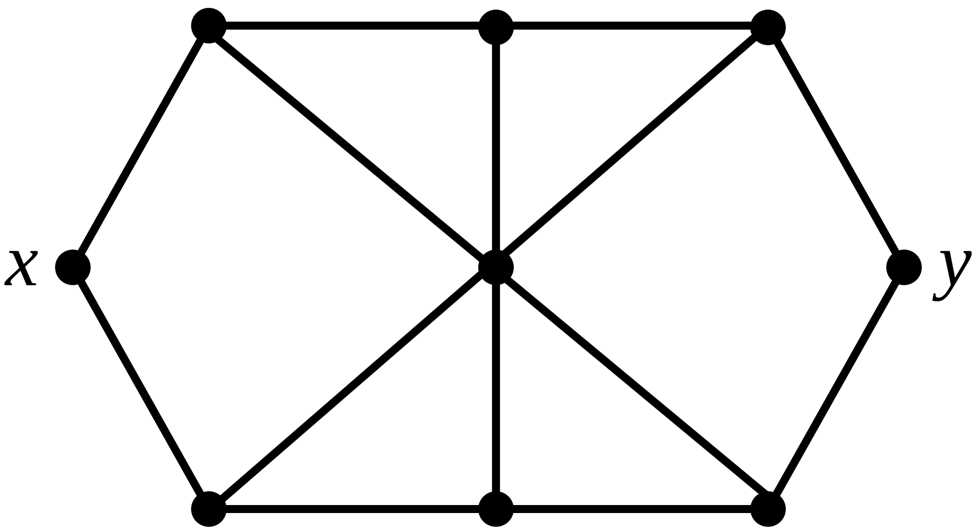
- 1. Main result
- 2. Applications
- 3. Kelmans operation

Terminology

- Degree: the degree of a vertex v in G is the number of its neighbors.
- Path: $P=(v_0,e_1,v_1,e_2,\ldots,e_k,v_k)$ where $e_i=v_{\mathrm{i-1}}v_i$ and $v_i \neq v_{\mathrm{j}}$ for $i \neq j$. An (x,y)-path is one from x to y.
- Cycle: $C=(v_0,e_1,v_1,e_2,\ldots,e_k,v_0)$
- Connectivity: G is connected if for each two vertices u, v of G, there is a (u,v)-path. G is k-connected if G-S is connected for any vertex set S with $|S|<k$.

Degree, path, cycle



Erdős-Gallai Theorem:

- **Erdős-Gallai [1]:**

If G is 2-connected, x, $y \in V(G)$, and every vertex other than x, y has degree at least k, then G has an (x, y)-path of length at least k.

[1] P. Erdős, T. Gallai, On maximal paths and circuits of graphs, Acta Math. Acad. Sci. Hungar. 10 (1959), 337–356.

Generalizations of Erdős-Gallai Theorem:

Erdős-Gallai Theorem has some other generalization, e.g. For Ore's degree sum condition, for Fan-type degree condition, for paths passing through given vertex set, for weighted graphs.

Enomoto [2]: If G is 2-connected, x, $z \in V(G)$, and every vertex other than x, z has degree at least k, then Then for any given vertex y of G, G has an (x, y, z)-path of length at least k.

[3] H. Enomoto, Long paths and large cycles in finite graphs, J. Graph Theory 8 (1984) 287-301.

Generalizations

Fan [3]: Let G be a 2-connected graph and x and y be two distinct vertices of G. If the average degree of the vertices other than x and y is at least k, then G contains an (x, y)-path of length at least k.

Bondy-Fan [4]: Let G be a 2-connected weighted graph and d a real number. Let x and z be two distinct vertices of G. If $d^w(v) \geq k$ for all $v \in V(G) \setminus \{x, y\}$, then G contains an (x, y)-path of weight at least k.

[3] G. Fan, Long cycles and the codiameter of a graph, I. J. Combin. Theory Ser B 49, 151–180 (1990).

[4] J.A. Bondy, G. Fan, Optimal paths and cycles in weighted graphs, Ann. Discrete Math. 41 (1989) 53{69.

Bondy-Jackson Theorem:

Set $n_k(x, y) = |\{z \in V(G) \backslash \{x, y\}: d(z) \geq k\}|$.

- Erdős-Gallai Theorem:

  $n_k(x, y) \geq n - 2 \rightarrow$ an (x, y)−path of length $\geq k$.

- Bondy-Jackson Theorem [5]:
- If G is 2-connected with $|V(G)|\geq 4$, x, $y \in V(G)$, and every vertex in $V(G)\backslash\{x, y\}$, with possibly one exception, has degree at least k, then G has an (x, y)-path of length at least k.

  $n_k(x, y) \geq n - 3 \rightarrow$ an (x, y)−path of length $\geq k$.

[5] J.A. Bondy, B. Jackson, Long paths between specified vertices of a block, Ann. Discrete Math. 27 (1985), 195–200.

Our Main Theorem

- It is natural to ask what is the best lower bound on $n_k(x,y)$ which still ensures the same conclusion.

Theorem 1 [6]:

- If G is 2-connected of order n, x, $y \in V(G)$, and there are at least $\frac{n-1}{2}$ vertex in $V(G) \backslash \{x,y\}$ has degree at least k, then G has an (x, y)-path of length at least k.

$$n_k(x,y) \geq \frac{n-1}{2} \rightarrow \text{an } (x, y)\text{–path of length} \geq k.$$

[6] B. Li, B. Ning, A strengthening of Erdős-Gallai Theorem and proof of Woodall's conjecture, J. Combin. Theory Ser B 146, 76-95 (2021).

Extremal graphs

- **Extremal graphs:**

- Let $k \geq 5$ be an odd integer, $H = K_{\frac{k-1}{2}} \vee \overline{K_{\frac{k-1}{2}}}$. Let G be from t disjoint copies of H, by adding two new vertices x and y followed by all edges between x, y and the $K_{\frac{k-1}{2}}$ subgraph of each copy of H.
- There are exactly $\frac{n-2}{2}$ vertices other than $\{x, y\}$ of degree at least k, and the length of a longest (x, y)-path $= k - 1$.

Dirac's Theorem:

- **Dirac [8]:** Let G be a 2-connected graph on n vertices. If every vertex of G has degree at least d, then G contains a cycle of length at least $\min\{2d, n\}$.
- Extremal Graphs: complete bipartite graph $K_{d-1,\, n-d+1}$.

[8] G.A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. (3) 2 (1952), 69--81

Generalization of Dirac's theorem:

- **Pósa [9]:** Let G be a 2-connected graph on n vertices. If the degree sum of every two non-adjacent vertices of G is at least $2d$, then G contains a cycle of length at least $\min\{2d, n\}$.
- **Fan [10]:** Let G be a 2-connected graph on n vertices. If every pair of vertices x, y with distance 2 satisfies that $\max\{d(x), d(y)\} \geq d$, then G contains a cycle of length at $\min\{2d, n\}$.

[9] L. Pósa, On the circuits of finite graphs. Magyar Tud. Akad. Mat. Kutató Int. Közl. 8 (1963) 355–361.

[10] G. Fan, New sufficient conditions for cycles in graphs, J. Combin. Theory Ser. B 37 (1984) 221–227.

Generalization of Dirac's theorem:

- **Erdős, Gallai [1]:** Let G be a graph on n vertices and m edges. If $m \geq n$, then G contains a cycle of length at least $\frac{2m}{n-1}$.
- **Locke [11]:** Let G be a 2-connected graph on n vertices. If every vertex of G has degree at least k, then for any two vertex x, y of G, G contains a cycle of length at least $\min\{2k, n\}$ passing through x, y.

[1] P. Erdős, T. Gallai, On maximal paths and circuits of graphs, Acta Math. Acad. Sci. Hungar. 10 (1959), 337–356.

[11] S.C. Locke, A generalization of Dirac's theorem, Combinatorics 5 (2) (1985) 149-159.

Woodall's Conjecture

- **Woodall's Conjecture [12]:**

Let G be a 2-connected graph on n vertices. If there are at least $\frac{n}{2} + k$ vertices of degree at least k, then $c(G) \geq 2k$.

[12] D. R. Woodall, Maximal circuits of graphs II, Studia Sci. Math. Hungar. 10 (1975), 103–109.

Woodall's Conjecture

- Häggkvist-Jackson [13]: $n \leq 3k - 2$ ($2k$ vertices of degree $\geq k$).
- Li-Li [14]: $n \leq 4k - 6$.
- Häggkvist-Li [15]: 3-connected, $k \geq 25$.
- Li [15]: Under the condition, $c(G) \geq 2k - 13$.
- Li: ($k \geq 683$).

[13] R. Häggkvist, B. Jackson, A note on maximal cycles in 2-connected graphs, Ann. Discrete Math. 27 (1985), 205–208.

[14] D. Li, H. Li, On longest cycles in graphs, Rapport de recherche no 1160, LRI, URA 410 du CNRS, Bat. 490, Univ de Paris sud, 91405-Orsay, France.

[15] H. Li, On a conjecture of Woodall, J. Combin. Theory Ser. B 86 (1) (2002) 172–185.

Woodall's conjecture

- **Theorem 2 [6]:**

Let G be a 2-connected graph on n vertices. If there are at least $\frac{n}{2}+k$ vertices of degree at least k, then $c(G) \geq 2k$.

[6] B. Li, B. Ning, A strengthening of Erdős-Gallai Theorem and proof of Woodall's conjecture, J. Combin. Theory Ser B 146, 76-95 (2021).

- 2. Applications

Fan lemma

Let G be a graph, C a cycle of G, and H a component of $G-C$. An **(*H*,*C*)-fan** consists of paths P_1, P_2, ..., P_t, $t \geq 2$, such that: (1) all P_i have the same origin $v \in V(H)$ and pairwise different termini $u_i \in V(C)$; (2) all internal vertices of P_i are in H and P_i's are pairwise internally disjoint.

Theorem (Fujisawa, Yoshimoto, Zhang[16]). Let G be a 2-connected graph, C a cycle of G, and H a component of $G - C$. If each vertex in H has degree at least k in G, then there is an (H, C)-fan with at least k edges.

[16] J. Fujisawa, K. Yoshimoto, S. Zhang, Heavy cycles passing through some specified vertices in weighted graphs, J. Graph Theory 49 (2005) (2), 93–103.

Woodall-type fan lemma

Theorem 3, Woodall-type Fan Lemma [4]:
Let G be a 2-connected graph, C a cycle of G, and H a component of $G-C$. If there are at least $\frac{|H|+1}{2}$ vertices in $V(H)$ of degree at least k in G, then there is an (H,C)-fan with at least k edges.

Lemma. Let G be a 2-connected non-hamiltonian graph, C a longest cycle and H a component of $G - C$. If there is an (H,C)-fan with at least k edges, then $|C| \geq 2k$.

Proof of Woodall's conjecture

$n/2+k$ vertices of degree $\geq k \to c(G) \geq 2k$

Let C be a longest cycle of G. By condition $n \geq 2k$. If G is Hamiltonian then $|C| \geq 2k$. So let H_i, $i = 1, \ldots, t$, be the components of $G - C$.

If H_i contains more than half vertices of degree $\geq k$, then an (H_i,C)-fan of edge number $\geq k$, then $|C| \geq 2k$.

So the number of vertices of degree $\geq k$ is at most

$$|C| + \sum_{i=1}^{t} \frac{|H_i|}{2} = \frac{n+|C|}{2}.$$

We have $\frac{n+|C|}{2} \geq \frac{n}{2} + k$ and $|C| \geq 2k$.

Bermond's conjecture

Bermond's Conjecture [17]. Let G be a 2-connected graph with vertex set $V = \{x_i : 1 \le i \le n\}$ and $c \le n$. If for any pair of vertices (x_i, x_j), $i < j$, one of the following holds:

(i) $i + j < c$; (ii) $x_i x_j \in E(G)$; (iii) $d(x_i) > i$;

(iv) $d(x_j) \ge j$; (v) $d(x_i) + d(x_j) \ge c$,

then $c(G) \ge c$.

Theorem 4 [6]. Bermond's conjecture is true for $n \ge 3c - 1$.

[17] J.C. Bermond, On Hamiltonian walks, Congr. Numer. 15 (1976) 41–51.

Bazgan-Li-Woźniak's theorem

Lobel, Komlos and Sós' conjecture: Suppose that G is an n-vertex graph with at least $n/2$ vertices of degree at least k. Then G contains each tree of order $k+1$.

- The following is a special case of Loebl-Komlós-Sós' conjecture

Theorem 7 (Bazgan, Li, Woźniak [18])
Let G be a graph on n vertices. If there are at least $\frac{n}{2}$ vertices of degree at least k, then G contains a path of length at least k.

[18] C. Bazgan, H. Li, M. Woźniak, On the Loebl-Komlós-Sós conjecture, J. Graph Theory 34 (2000), no. 4, 269–276.

Some theorems on cycles

- **Theorem** (Dirac [8]). Let G be a 2-connected graph on n vertices. If the degree of every vertex is at least k, then $c(G) \geq \min\{n, 2k\}$.
- **Theorem** (Erdős, Gallai [1], Bondy [19]). Let G be a 2-connected graph on n vertices. If the degree of every vertex other than one is at least k, then $c(G) \geq \min\{n, 2k\}$.

[8] G.A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. 2 (1952) 69–81.

[19] J.A. Bondy, Large cycles in graphs, Discrete Math. 1 (1971/72), no. 2, 121–132.

Some theorems on cycles

- **Theorem 8 [6]**. Let G be a 2-connected graph on $n \geq 2ks + 3$ vertices and $x, y \in V(G)$. If $\max\{d(v): v \in S\} \geq k$ for any independent set $S \subset V(G) \setminus \{x,y\}$ with $|S|=s+1$, then G has an (x,y)-path of length at least k.

- **Theorem 9 [6].** Let G be a 2-connected graph on $n \geq 2k(s+1)$ vertices. If $\max\{d(v): v \in S\} \geq k$ for any independent set $S \subset V(G)$ with $|S|=s+1$, then $c(G) \geq 2k$.

Bondy's conjecture

- **Theorem** (Fournier, Fraisse [20]). Let G be an s-connected graph on n vertices where $s \geq 2$. If the degree sum of any independent set of size $s + 1$ is at least m, then $c(G) \geq \min\left\{\frac{2m}{s+1}, n\right\}$.

[20] I. Fournier, P. Fraisse, On a conjecture of Bondy, J. Combin. Theory Ser. B 39 (1985), no. 1, 17–26.

Häggkvist-Jackson's conjecture

Häggkvist-Jackson's conjecture [21]

Let G be a 2-connected graph on n vertices. If G contains at least $\max\left\{2k-1, \frac{n+k}{2}+1\right\}$ vertices of degree at least k, then G has a cycle of length at least $\min\{n, 2k\}$.

[21] R. Häggkvist, B. Jackson, A note on maximal cycles in 2-connected graphs, Ann. Discrete Math. 27 (1985), 205–208.

Häggkvist-Jackson's conjecture

- Let $G_1 = K_2 \vee (K_{2k-4} + \overline{K_t})$.
- Let $H_1 = K_{\frac{k-1}{2}} \vee \overline{K_{\frac{k-1}{2}}}$, $H_2 = K_{k+1}$ ($k \geq 3$ is odd). Let G_2 be the graph obtained from one copy of H_2 and t disjoint copies of H_1 by joining each vertex in the $K_{\frac{k-1}{2}}$ subgraph of H_1 to two fixed vertices of H_2.
- One can see G_1 has $2k-2$ vertices of degree at least k, G_2 has $\frac{n+k+1}{2}$ vertices of degree at least k, and $c(G_1) = c(G_2) = 2k-1$.

Li's conjecture

- Li [21] conjectured that for any 2-connected graph G of order n, there is a cycle of length at least $2k$ if the number of vertices of degree at least k is at least $\frac{n+k}{2}$. The constructions G_1 and G_2 mentioned above disprove Li's conjecture.

[21] H. Li, Generalizations of Dirac's theorem in Hamiltonian graph theory–a survey, Discrete Math. 313 (2013), no. 19, 2034–2053.

A conjecture

- We suggest the following conjecture, which is a generalization of Theorem 1.
- **Conjecture 1 [6]**. Let G be a 2-connected graph on n vertices and $x,y \in V(G)$. Let $0<\alpha\leq 1/2$. If $V(G)\backslash\{x,y\}$ contains more than $\alpha(n-2)$ vertices of degree at least k, then G contains an (x,y)-path of length at least $2\alpha k$.

A conjecture

- When α is a rational number, we choose k such that $\alpha(k-1)$ is an integer. Let $H = K_{\alpha(k-1)} \vee \overline{K_{(1-\alpha)(k-1)}}$. Let G be obtained from t copies of H, by adding two new vertices $\{x, y\}$ and all possible edges between $\{x,y\}$ and the $K_{\alpha(k-1)}$ subgraph of each H. The number of vertices of degree at least k in G is $\alpha(|G|-2)$ and a longest (x,y)-path is of length $2\alpha(k-1)$.
- This example shows that Conjecture 1 is sharp for infinite values of integers n and k.

- 3. Kelmans operation

Kelmans operation

- Let G be a graph and $u,v \in V(G)$. A new graph $G' := G[u \to v]$ is a **Kelmans graph** [22] of G (from u to v), if $V(G')=V(G)$ and $E(G') = (E(G) \setminus \{uw: w \in N[u] \setminus \mathrm{N}[v]\}) \cup \{vw: w \in N[u] \setminus \mathrm{N}[v]\}$.

[22] A.K. Kelmans, On graphs with randomly deleted edges, Acta Math. Acad. Sci. Hungar. 37 (1981)77–88.

Order of degree sequences

- Assume $\tau(G) = (d_1, d_2, \ldots, d_n)$, $\tau(G') = (d'_1, d'_2, \ldots, d'_n)$ are non-increasing degree sequences of G and G', respectively. If there exists an integer j such that $d_i = d'_i$ for $1 \le i \le j-1$ and $d_j = d'_j$, then we say that $\tau(G)$ is larger than $\tau(G')$ and denote it by $\tau(G) > \tau(G')$.

A lemma on Kelmans operation

- **Lemma [6].** Let G be a graph, x, y, u be distinct vertices of G, and $v \in N(u)$ (possibly $v \in \{x, y\}$). Let $G' = G[u \to v]$.

  (i) If neither $N[u] \subseteq N[v]$ nor $N[v] \subseteq N[u]$, then $\tau(G') > \tau(G)$.

  (ii) If G' has an (x, y)-path of length at least k, then so does G.

Thanks!